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# LOWER BOUNDS FOR COVERINGS OF PAIRS BY LARGE BLOCKS

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Let $n \geqq k \geqq t$ be positive integers, and $X$ — a set of $n$ elements. Let $C(n, k, t)$ be the smallest integer $m$ such that there exist $m$ $k$-tuples of $X$ $B_1, B_2, \ldots, B_m$ with the property that every $t$-tuple of $X$ is contained in at least one $B_i$. It is shown that in many cases the standard lower bound for $C(n, k, 2)$ can be improved ($k$ sufficiently large, $n/k$ being fixed). Some exact values of $C(n, k, 2)$ are also obtained.
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## 1. Introduction

Let $n \geqq k \geqq t$ be positive integers, and let $X$ be a set of $n$ elements, i.e. $|X|=n$. A family $F=\{B_1, \ldots, B_m\}$ of $k$-tuples of $X$ (subsets of $X$ having $k$ elements each) is called $(n, k, t)$-covering if every $t$-tuple of $X$ is contained in at least one $B_i$. The elements of $F$ are called blocks. $C(n, k, t)$ denotes the smallest integer $m$ such that there exists an $(n, k, t)$-covering $F$ (called covering design, optimal covering etc.) having $m$ blocks.

Let $\alpha \in X$, $G \subseteq F$. By $G(\alpha)$ denote the number of blocks in $G$ containing $\alpha$, i.e., $G(\alpha)=|\{B \in G\colon \alpha \in B\}|$. In the case $G=F$ it is easy to see that $F(\alpha) \geqq C(n-1, k-1, t-1)$ and then

$$(1) \qquad C(n, k, t) \geqq \lceil nC(n-1, k-1, t-1)/k \rceil,$$

where $\lceil x \rceil$ denotes the smallest integer that is at least $x$.

Iterating (1) we obtain

$$(2) \qquad C(n, k, t) \geqq \left\lceil \frac{n}{k} \left\lceil \frac{n-1}{k-1} \left\lceil \ldots \left\lceil \frac{n-t+1}{k-t+1} \right\rceil \ldots \right\rceil \right\rceil \right\rceil = L(n, k, t),$$

which is the Schönheim lower bound for $C(n, k, t)$ [1, 2]. There are many cases in which the lower bound (2) is actually achieved [1, 2, 3, 4, 7]. For example, the well known Turán's theorem [3, 4] gives $C(n, n-2, t)=L(n, n-2, t)$. It was conjectured by Erdős and Hanani [11] and proved by Rödl [9] that for $k$, $t$ fixed

$$(3) \qquad \lim_{n \to \infty} C(n, k, t)/L(n, k, t) = 1.$$

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It follows from the results of Kuzjurin [10] that (3) holds for $t$ fixed and for arbitrary sequence $k=k(n)$ with $\lim_{n\to\infty} k/n=0$. So, the lower bound (2) is a good approximation for $C(n, k, t)$ when $t$ is fixed. Some possibilities to improve (2) are studied in [5, 6]. It follows from (1) that if $C(n-1, k-1, t-1)>L(n-1, k-1, t-1)$ then $C(n, k, t)>L(n, k, t)$. So, the improvement of the Schönheim lower bound for $t=2$ leads to an improvement for arbitrary $t$. For $t=2$ and $k$ fixed it seems that for all sufficiently large $n$ we have $C(n, k, 2)=L(n, k, 2)$ except the case $n\equiv 1 \pmod{k-1}$ $n(n-1)\equiv 1 \pmod{k}$ (Mills [5]). Here we consider some possibilities to improve the lower bound (2) for $t=2$ and for sufficiently large $k$.

Let $F$ be an $(n, k, 2)$-covering and $\alpha, \beta\in X$. Let $\mathrm{cl}\,\alpha=\{B\in F:\ \alpha\in B\}$, $F(\alpha, \beta)=|\{B\in F:\ \{\alpha, \beta\}\subset B\}|$, $\mathrm{Br}(\alpha)=\{\beta\in X:\ F(\alpha, \beta)>1\}$. Clearly $F(\alpha)=|\mathrm{cl}\,\alpha|$. Since $C(n-1, k-1, 1)=\lceil (n-1)/(k-1)\rceil$ then $F(\alpha)\geqq\lceil (n-1)/(k-1)\rceil$. Denote $X_1=\{\alpha\in X:\ F(\alpha)=\lceil (n-1)/(k-1)\rceil\}$, $X_2=X\setminus X_1$. The elements of $X_1$ are called *centres*, and those of $X_2$ *non-centres*. A block $B\in F$ is said to be an $i$-block if $|X_2\cap B|=i$. Since

$$k|F| = \sum_{\alpha\in X} F(\alpha)$$

then

$$|F| \geqq (n\lceil (n-1)/(k-1)\rceil+|X_2|)/k. \tag{4}$$

Let $n-1=s(k-1)-j$, $0\leqq j<k-1$, $B\in F$, $B\cap X_1=\{\alpha_1, \dots, \alpha_r\}$. Denote

$$\omega(B) = \sum_{1\leqq i<p\leqq r} \big(F(\alpha_i, \alpha_p)-1\big), \quad \varepsilon(\alpha) = \sum_{x\in X\setminus\{\alpha\}} \big(F(\alpha, x)-1\big).$$

For every $\alpha\in X_1$ we have $F(\alpha)=s$, and $\varepsilon(\alpha)=s(k-1)-n+1=j$. Therefore

$$2\omega(B) \leqq jr. \tag{5}$$

Note that if $2\omega(B)=jr$ then $\mathrm{Br}(\alpha_i)\subset B\cap X_1$ for every $i=1, \dots, r$.

Finally, denote by $\mathrm{SF}(\alpha)$ the set of all $\beta\in X$ which are contained in excatly the same blocks of $F$ as $\alpha$ $\big(\alpha\in \mathrm{SF}(\alpha)\big)$.

## 2. The *projective* zone

It was proved in [6] that if $q$ is a prime power, and $a_0\geqq a_1\geqq a_2>0$ are integers then for $k=a_0q+a_1$, $n=a_0q^2+a_1q+a_2$:

$$C(n, k, 2) = L(n, k, 2) = q^2+q+1.$$

The optimal coverings were obtained from the projective plane $\mathrm{PG}(2, q)$. It seems that the condition $a_0\geqq a_1\geqq a_2$ is essential. More exactly, we suppose that if $k\geqq s+1$, $s<n/k\leqq s+1/(s+1)$ (the *projective* zone), and there exist no integers $a_0\geqq a_1\geqq a_2$ such that $k=a_0s+a_1$, $n=a_0s^2+a_1s+a_2$ then

$$C(n, k, 2) > L(n, k, 2) = s^2+s+1. \tag{6}$$

For $s=2$ (6) holds (see [5]). It will also hold for $s=3$ if $C(28, 9, 2)>13$ and $C(41, 13, 2)>13$ [5, 8]. The validity of (6) will give a number of exact values of $C(n, k, 2)$ in the projective zone. For example, let $q$ be a prime power. Consider a line $B=\{b_0, b_1, \dots, b_q\}$ in $\mathrm{PG}(2, q)$ and substitute every point $b_i$, $i=1, \dots, q$

with $q$ new points $\{b_i^1, ..., b_i^q\}$. Replace $B$ with two new blocks $B'=B'\setminus\{b_1^1\}$, $B''=$ $=B\setminus\{b_1^2\}$. Now substitute every point $a\notin B$ with $q-1$ new points. The lines in PG $(2, q)$ are transformed into blocks of $(q^3+1, q^2, 2)$-covering (some of them with cardinality less than $q^2$) which yields $C(q^3+1, q^2, 2)\leqq q^2+q+2$, and if (6) holds then $C(q^3+1, q^2, 2)=q^2+q+2$. Here we shall prove (6) in the following special case.

**Theorem 2.1.** *If* $n=s^2+as+1$, $k=s+a$, $s\geqq a>1$ *then*

$$C(n, k, 2) > L(n, k, 2) = s^2+s+1.$$

**Proof.** Suppose that there exists an $(n, k, 2)$-covering $F=\{B_1, ..., B_m\}$ with $m=s^2+$ $+s+1$. For $\alpha\in X_1$ we have $F(\alpha)=s+1$, $\varepsilon(\alpha)=a-1$, and from (4) $|X_2|\leqq a-1$. Since

$$\sum_{\alpha\in X_2} F(\alpha) = k|F| - \sum_{\alpha\in X_1} F(\alpha) \leqq s^2+s-2$$

there exists a 0-block $B_1\in F$ $(B_1\cap X_2=\emptyset)$. From (5) $2\omega(B_1)\leqq(a-1)(s+a)$. On the other hand

$$2\omega(B_1) = \sum_{i=2}^{m} \omega_i(\omega_i-1)$$

where $\omega_i=|B_1\cap B_i|$, $i\geqq 2$. Thus

$$2\omega(B_1) = \sum_{i=2}^{m} \omega_i^2 - \sum_{i=2}^{m} \omega_i = \sum_{i=2}^{m} \omega_i^2 - s(s+a).$$

Let $\bar\omega=\max_i \omega_i$, $\underline{\omega}=\min_i \omega_i$. The minimum of the last sum is achieved for $\bar\omega-\underline{\omega}\leqq 1$, and consequently $\underline{\omega}\geqq 1$, $\bar\omega\leqq 2$. So

$$\sum_{i=2}^{m} \omega_i^2 \geqq 3as+s^2-2s.$$

Therefore

(7) $$(s+a)(a-1) \geqq 2\omega(B_1) \geqq 2s(a-1)$$

which result in a contradiction if $a<s$. If $a=s$ (7) gives $2\omega(B_1)=2s(s-1)$, $1\leqq$ $\leqq\omega_i\leqq 2$, $i\geqq 2$, and $\mathrm{Br}(\alpha)\subset B_1$ for every $\alpha\in B_1$. Let $\xi\in B_1$, $\mathrm{cl}\,\xi=\{B_1, B_2, ..., B_{s+1}\}$. Since $\mathrm{Br}(\xi)\subset B_1$ and $|X_2|\leqq s-1$ then there is another 0-block in $\mathrm{cl}\,\xi$, say $B_2$. According to the above $\mathrm{Br}(\xi)\subset B_1\cap B_2$, $|B_1\cap B_2|=2$. Therefore $|\mathrm{Br}(\xi)|=1$. Suppose that $\mathrm{Br}(\xi)=\{\eta\}$. Obviously $F(\xi, \eta)=s$ and let $\mathrm{cl}\,\eta=\{B_1, ..., B_s, B_{s+2}\}$. We have to cover the pairs $\{\eta, B_{s+1}\setminus\{\xi\}\}$ and then $B_{s+1}\setminus\{\xi\}=B_{s+2}\setminus\{\eta\}$ which yields $B_{s+1}\setminus\{\xi\}\subset X_2$ and consequently $|X_2|\geqq 2s-1$, a contradiction. ∎

## 3. The case $n-1\equiv 0 \pmod{k-1}$

The theorem below presents an extension of the [5, 6] results concerning the case $n-1\equiv 0 \pmod{k-1}$. Since $C(n, k, 2)$, $n\leqq 3k$ is completely known [5], then we shall assume that $n>3k$.

**Theorem 3.1.** Let $n-1=s(k-1)$, $s>3$.

(A) If $k>s$ then $C(n,k,2)\geqq s^2+s-2s(s-1)/k$.

(B) If $n(n-1)\equiv j \pmod k$, $1<j<k$, $s<j(k-1)/(j-1)$ then

$$C(n,k,2)>L(n,k,2).$$

**Proof.** (A) Let $F=\{B_1,\dots,B_m\}$ be an $(n,k,2)$-covering with $n$, $k$ satisfying the conditions of the theorem. Clearly $F(\alpha)=s$ for every $\alpha\in X_1$. It was shown in [6] (Theorem 1) that $|X_1|\leqq(s-1)^2$, or there exists a block $B\in F$ such that $X_1\subseteq B$. In the first of these cases we have $|F|\geqq(ns+n-(s-1)^2)/k=s^2+s-2s(s-1)/k$ due to (4). Now suppose that $X_1\subseteq B_1$, $|X_1|>(s-1)^2$. Since $\varepsilon(\alpha)=0$ for every $\alpha\in X_1$ then $\omega(B_1)=0$ due to (5), and $|B_1\cap B_i\cap X_1|\leqq 1$, $i\geqq 2$. Thus $|F|\geqq|X_1|(s-1)+1>$ $>(s-1)^3+1>s^2+s-2s(s-1)/k$.

(B) If $k>s$ then the assertion follows from (A). If $s=k, k+1$ then $j=0$. Therefore we can assume that $s\geqq k+2$.

Let $F$ be an $(n,k,2)$-covering, $|F|=L(n,k,2)$. Since $n(n-1)\equiv j \pmod k$ then $s(s-1)\equiv j \pmod k$ and $|F|=\lceil ns/k\rceil=\lceil(s^2k-s(s-1))/k\rceil=(ns+j)/k$. Therefore $|X_2|\leqq j$. Every pair $(\alpha,\beta)$, $\alpha\in X_1$ is contained in exactly one block of $F$ which yields $|X_2|\geqq 2$ (if $X_2=\{\omega\}$ then there is a pair $(\alpha,\omega)$ with $F(\alpha,\omega)>1$). Note that if $j=1$ this gives $C(n,k,2)>L(n,k,2)$ [5]. In order to clear the idea of the proof for arbitrary $j>1$ first consider the case $j=2$. If $X_2=\{\omega_1,\omega_2\}$ then $F(\omega_1)=$ $=s+1$, $F(\omega_2)=s+1$, $F(\omega_1,\omega_2)=k$. Hence there is a block $B$ in $\operatorname{cl}\omega_1$ with $\omega_2\notin B$. We have to cover the pairs $(\alpha,\omega_2)$, $\alpha\in B\setminus\{\omega_1\}$ in different blocks of $\operatorname{cl}\omega_2\setminus\operatorname{cl}\omega_1$. So $s+1=F(\omega_2)=k+|\operatorname{cl}\omega_2\setminus\operatorname{cl}\omega_1|\geqq k+k-1=2k-1$, a contradiction. Thus we can assume that $j\geqq 3$, which yields $s\geqq k+3$ (if $s=k+2$ then $j=2$) and consequently $k\geqq 4j-2$.

Let $|X_2|=p>1$, and $\{\alpha_1,\dots,\alpha_q\}\subseteq X_2$, $X_2=\bigcup_{i=1}^{q}\mathrm{SF}(\alpha_i)$, $\mathrm{SF}(\alpha_i)\cap\mathrm{SF}(\alpha_f)=\emptyset$, $1\leqq i<f\leqq q$, $|\mathrm{SF}(\alpha_i)|=\beta_i$, $F(\alpha_i)=s+x_i$, $x_i>0$, $i=1,\dots,q$. Thus

$$\sum_{i=1}^{q}\beta_i=p \quad\text{and}\quad \sum_{i=1}^{q}x_i\beta_i=j.$$

Since $\varepsilon(\alpha)=0$ for $\alpha\in X_1$, and $|X_1|=s(k-1)+1-p$ then

(8)
$$\sum_{B\in\operatorname{cl}\alpha_i}|X_2\cap B|=x_ik+s+p-1.$$

Since $\sum_{B\in\operatorname{cl}\alpha_i}|X_2\cap B|\geqq(s+x_i)\beta_i+p-\beta_i$ then $\beta_i\leqq x_i$, $i=1,\dots,q$. Now consider $\operatorname{cl}\alpha_1$, and let $B$ be a $z$-block in $\operatorname{cl}\alpha_1$. We shall prove that:

(9)
$$-z^2+z(2p-1)\geqq(x_1-\beta_1)(4j-2)+3j-2p+\beta_1(p-1).$$

Let $\{\alpha_1,\dots,\alpha_h\}\subseteq B\cap X_2$, $\sum_{i=1}^{h}\beta_i=z$, $F(\alpha_1,\alpha_i)=\omega_i$, $i=2,\dots,h$, $F(\alpha_1,\alpha_i)=y_i$, $i=h+1,\dots,q$. We have to cover the pairs $(\alpha,\alpha_i)$, $\alpha\in B\cap X_1$, $i\geqq h+1$ and since $\alpha_i\notin B$, $i\geqq h+1$ then

(10)
$$y_i\leqq s+x_i-k+z.$$

Now:

$$\sum_{B\in \mathrm{cl}\,\alpha_1} |X_2\cap B| = \beta_1(s+x_1)+\sum_{i=2}^{h}\beta_i\omega_i+\sum_{i=h+1}^{q}\beta_i y_i,$$

and combining with (8) and (10) we get

$$(11)\qquad \sum_{i=2}^{h}\beta_i\omega_i \geqq x_1k+s+p-1-\beta_1(s+x_1)-(p-z)(s+z-k)-\sum_{i=h+1}^{q}x_i\beta_i.$$

On the other hand if $\alpha_i\notin B_f$ $(B_f\in \mathrm{cl}\,\alpha_1)$ then $|B_f\cap X_1|\leqq s+x_i-\omega_i$, i.e. $|B_f\cap X_2|\geqq$ $\geqq k-(s+x_i-\omega_i)$. Hence $k-(s+x_i-\omega_i)\leqq p-\beta_i$ which yields

$$(12)\qquad \beta_i\omega_i \leqq \beta_i(s+p-k+x_i-\beta_i),\quad i=2,\dots,q.$$

Summing (12) over $i=2,\dots,\alpha$ and using that

$$\sum_{i=2}^{h}\beta_i = z-\beta_1,\quad \sum_{i=2}^{n}\beta_i^2 \geqq z-\beta_1$$

we obtain

$$(13)\qquad \sum_{i=2}^{h}\beta_i\omega_i \leqq (z-\beta_1)(s+p-k-1)+\sum_{i=2}^{h}x_i\beta_i,$$

(11) and (13) give

$$-z^2+z(2p-1)\geqq k(p+x_1-\beta_1)-(p-1)(s-\beta_1-1)-j.$$

Using that $s<j(k-1)/(j-1)$, and $k\geqq 4j-2$ we obtain (9). It follows from (9) that there are no 1-blocks in $F$, i.e. $z>1$.

Let $x_1=\min_i x_i$. We shall prove that every block in $\mathrm{cl}\,\alpha_1$ contains at least $x_1+1$ non-centres. The maximal value $p(p-1)$ of the left side in (9) is achieved for $z=p$, $p-1$. Using in addition that $\beta_1\leqq p$ we get $x_1\leqq p-1$. Now suppose that there exists a $z$-block in $\mathrm{cl}\,\alpha_1$ with $z\leqq x_1$. Since $z\leqq x_1\leqq p-1$ we can substitute $z$ by $x_1$ in (9). This gives

$$(14)\qquad x_1p\geqq (x_1-\beta_1)(4j-p-1)+3j-2p+x_1^2.$$

Since $x_1=\min x_i$ then $j\geqq x_1p$ and (14) is impossible. Thus for every $z$-block in $\mathrm{cl}\,\alpha_1$ $z\geqq x_1+1$. If in addition $z\geqq x_1+2$ then

$$\sum_{B\in \mathrm{cl}\,\alpha_1}|X_2\cap B|\geqq (x_1+2)(s+x_1)$$

which is impossible due to (8). So, there exists a $(x_1+1)$-block in $\mathrm{cl}\,\alpha_1$ and from (9) $(z=x_1+1)$:

$$x_1p\geqq (x_1-\beta_1)(4j-p-1)+3j-4p+1+(x_1+1)^2$$

which shows that $x_1=1$ (if $x_1\geqq 2$ then $p\leqq j/2$ since $x_1p\leqq j$). Therefore $\beta_1=1$ $(x_1\geqq\beta_1)$. Suppose that $B_1\in \mathrm{cl}\,\alpha_1$ is a 2-block and $B_1\cap X_2=\{\alpha_1,\alpha_2\}$. Hence $\beta_2=1$ and since there exists a 2-block in $\mathrm{cl}\,\alpha_2$ then we can make use of (9) substituting $x_1$ by $x_2$, $\beta_1$ by 1, $z$ by 2. Thus $0\geqq(4j-2)x_2-5p-j+7$ and consequently $x_2=1$.

The next step is to show that there are 2-blocks in $\mathrm{cl}\,\alpha_1$ with different non-centres. Denoting the number of 2-blocks in $\mathrm{cl}\,\alpha_1$ by $\tau$ we have $2\tau+3(s+1-\tau)\leqq$

$\leqq k+s+p-1$. Therefore

$$\tau \geqq 2s-k-p+4>0. \tag{15}$$

Suppose that every 2-block in $\mathrm{cl}\,\alpha_1$ contains $\alpha_2$. This gives $F(\alpha_1, \alpha_2) \geqq 2s-k-p+4$ due to (15). Thus if $B \in \mathrm{cl}\,\alpha_1 \setminus \mathrm{cl}\,\alpha_2$ then $|B \cap X_1| \leqq s+1-F(\alpha_1, \alpha_2) \leqq k-s+p-3$ (we have to cover the pairs $(\alpha, \alpha_2)$, $\alpha \in B \cap X_1$). Therefore $|B \cap X_2| \geqq k-(k-s+p-3)>p$, a contradiction.

Suppose that $B_0$, $B_1$ are 2-blocks in $\mathrm{cl}\,\alpha_1$, $\{\alpha_1, \alpha_2\}=B_1 \cap X_2$, $\{\alpha_1, \alpha_3\}=B_0 \cap X_2$. The pairs $(\alpha, \alpha_2)$, $\alpha \in B_0 \cap X_1$ are covered in different blocks of $\mathrm{cl}\,\alpha_2 \setminus \mathrm{cl}\,\alpha_1$. Hence

$$F(\alpha_1, \alpha_2) \leqq s+1-(k-2)=s-k+3 \leqq p. \tag{16}$$

The last inequality in (16) follows from (8) and from the fact that every block in $\mathrm{cl}\,\alpha_1$ contains at least 2 non-centres, i.e. $2(s+1) \leqq k+s+p-1$.

Let $\alpha \in B_1 \cap X_1$ and let $B_1, \ldots, B_r$ be $i$-blocks ($i \geqq 2$) of $\mathrm{cl}\,\alpha$ (there are no 1-blocks in $F$). Obviously $2 \leqq r \leqq p/2$. Let $\mathrm{cl}\,\alpha_1=\{B_1, B_2', \ldots, B_{s+1}'\}$. Denote

$$G=\{B_1' \in \mathrm{cl}\,\alpha_1\colon B_i' \cap B_f \cap X_1 \neq \emptyset,\ f=2, \ldots, r\}.$$

If $B_i' \in G$ then $B_i' \cap X_2=\{\alpha_1, \alpha_2\}$ since in the opposite case we can find a pair $(x, y)$, $x \in X_1$, $y \in X_2$ which is covered twice. Thus $F(\alpha_1, \alpha_2) \geqq |G|+1$. Let $w_i=|B_i \cap X_2|$, $i=2, \ldots, r$. Therefore

$$|G| \geqq s-\sum_{i=2}^{r}\bigl(s-(k-w_i-1)\bigr)=s-(r-1)(s-k+1)-(p-2) \geqq$$

$$\geqq -(r-2)j(k-1)/(j-1)+(r-1)(k-1)-j+2>j+6.$$

Thus $F(\alpha_1, \alpha_2) \geqq j+8$ which contradicts (16). ∎

There are many values of $n$, $k$ satisfying the conditions of Theorem 3.1 (B). For example $n=(k+m)(k-1)+1$, $k>m(m^2-2)$, $m \geqq 2$.

For sufficiently large $k$ Theorem 3.1 (A) gives sometimes the best possible.

**Corollary 3.2.** *Let $q$ be a prime power, $n-1=q(k-1)$, $q>3$, $k=qr$, $k>2q(q-1)$. Then for $n \leqq n' \leqq q^2 r$ we have*

$$C(n', k, 2)=C(n, k, 2)=L(n, k, 2)+q=q^2+q.$$

**Proof.** Since $q$ is a prime power then there exists an affine plane AG $(2, q)$. AG $(2, q)$ is an optimal $(q^2, q, 2)$-covering having $q^2+q$ blocks (the lines in AG $(2, q)$). Since $C(nf, kf, t) \leqq C(n, k, t)$ [5] and $C(n, k, t) \leqq C(n', k, t)$ for $n \leqq n'$ then

$$C(n, k, 2) \leqq C(n', k, 2) \leqq C(q^2 r, qr, 2) \leqq C(q^2, q, 2)=q^2+q.$$

The opposite inequalities follow from Theorem 3.1 (A). ∎

## 4. Coverings by large blocks

Let $n-1=q(k-1)-j$, $0<j<k-1$, $q>2$ and let $F=\{B_1, \dots, B_m\}$ be an $(n, k, 2)$-covering over an $n$-element set $X$. Clearly for every $\alpha \in X_1$ $F(\alpha)=q$, $\varepsilon(\alpha)=j$. The following lemmas are useful establish lower bounds for $C(n, k, 2)$ for sufficiently large values of $k$.

**Lemma 4.1.** *Let* $B\in F$, $q>i\geqq 0$, $|X_1\cap B|>(j+1)(q+i-1)$, $k>(j+1)(q+i)$, *and* $\xi\in X\setminus B$. *Then* $F(\xi)>q+i$.

**Proof.** Suppose that $F(\xi)=q+s$, $s\leqq i$, $\mathrm{cl}\,\xi=\{B_1, \dots, B_{q+s}\}$. Since $|B_r\cap B\cap X_1|\leqq j+1$, $r\leqq q+s$ and $|X_1\cap B|>(j+1)(q+i-1)$ then $B_r\cap B\cap X_1\neq\emptyset$ for every $r=1, \dots, q+s$. On the other hand $k>(j+1)(q+i)$, $B\subset\bigcup_{r=1}^{q+s} B_r$ and then we can assume that $|B_1\cap B|>j+1$. Hence $B_1\cap B\cap X_1=\emptyset$, a contradiction. ▮

We shall say that $B\in F$ is a maximal block if $|B\cap X_1|\geqq|B_i\cap X_1|$ for every other block $B_i\in F$.

**Lemma 4.2.** *Let* $B\in F$ *be a maximal block. Let i be integer,* $q>i>0$; $k>(j+1)q$ *and* $|B\cap X_1|>(j+1)(q-1)-ij(j+1)$. *Suppose that there exists an element* $\xi\in X_1\setminus B$, $\mathrm{cl}\,\xi=\{B_1, \dots, B_i, \dots, B_q\}$ *such that* $|B_r\cap B\cap X_1|=j+1$, $r=1, \dots, i$. *Then*

$$|X_1|\leqq (j+1)(iq-2i+q-1).$$

**Proof.** First we note that since $k>(j+1)q$ and there exists a centre $\xi\notin B$ then $|B\cap X_1|\leqq(j+1)(q-1)$ due to Lemma 4.1. Moreover, for every centre $\eta\notin B$ there exists a block $B'\in\mathrm{cl}\,\eta$ for which $|B'\cap B|>j+1$ and consequently $B'$ contains no centres from $B$. We shall prove that $X_1\subseteqq B\cup B_1\cup\dots\cup B_i=M$. Conversely, let $\eta\in X_1\setminus M$, $\mathrm{cl}\,\eta=\{B_{s_1}, \dots, B_{s_q}\}$. Without loss of generality we can assume that $B_{s_q}\cap B\cap X_1=\emptyset$, $|B_{s_h}\cap B\cap X_1|=1$ for $h=1, \dots, i(j+1)$ (if $i(j+1)>q-1$ this is impossible and such an $\eta$ does not exist). Since $|B_{s_h}\cap B\cap X_1|\leqq j+1$, $h=i(j+1)+1, \dots, q-1$ then

$$(j+1)(q-1-i(j+1))\geqq\Big|(B\cap X_1)\setminus\Big(\bigcup_{r=1}^{i} B_r\Big)\Big|>(j+1)(q-1)-ij(j+1)-i(j+1)$$

which is a contradiction. Thus $X_1\subseteqq M$ and

$$|X_1|\leqq(i+1)(j+1)(q-1)-i(j+1)=(j+1)(iq-2i+q-1). \ ▮$$

Further on, denote $\omega=\lceil (j+1)/2\rceil+1$. Let $B\in F$ be a maximal block and $\xi\in X_1\setminus B$, $\mathrm{cl}\,\xi=\{B_1, \dots, B_q\}$, $B_i^\xi=B_i\cap B\cap X_1$, $i=1, \dots, q$. Let $m(\xi)$ be the number of those $B_i^\xi$ for which $|B_i^\xi|\geqq\omega$. We assume that $|B_i^\xi|\geqq\omega$ for $i=1, \dots, m(\xi)$. Now if $1\leqq r_1<r_2\leqq m(\xi)$ then $B_{r_1}^\xi\cap B_{r_2}^\xi=\emptyset$ (if $\eta\in B_{r_1}^\xi\cap B_{r_2}^\xi$ then $\varepsilon(\eta)>j$ and consequently $\eta\in X_2$). So, every centre $\xi\in X_1\setminus B$ defines $m(\xi)$ disjoint subsets of $B\cap X_1$ with cardinality $\geqq\omega$. Denote $A(\xi)=\{B_1^\xi, B_2^\xi, \dots, B_{m(\xi)}^\xi\}$. Clearly if $B_i^\xi\in A(\xi)$, $B_s^\xi\in A(\eta)$ then either $B_i^\xi=B_s^\xi$ or $B_i^\xi\cap B_s^\xi=\emptyset$ due to the choise of $\omega$. Let $m^*=\min\{m(\xi)\colon \xi\in X_1\setminus B\}$ and

$$\bigcup_{\xi\in X_1\setminus B} A(\xi)=\{S_1, \dots, S_f\},\ S_i\subset B_i,\quad i=1, \dots, f,\ G=\{B_1, \dots, B_f\}.$$

**Lemma 4.3.** *Let* $|X_1\cap B|=\alpha q+\beta\geqq q(\omega-1)+1,\ 0\leqq\beta<q,\ X_1\setminus B\neq\emptyset,$ *B being a maximal block. Then*

$$|X_1|\leqq\min\{n-q(k-\alpha q-\beta)+j-\varrho+1,\ d(\alpha,\beta)\},$$

*where* $\varrho=\max_{\gamma,\delta x\in X_1}F(\gamma,\delta)$, $d(\alpha,\beta)=((m^*+f)(\alpha q+\beta)-f\tau)/m^*$, $\tau=\max\{\omega,\alpha\}$.

**Proof.** Obviously $G(\xi)\geqq m^*$ for every $\xi\in X_1\setminus B$ ($G(\xi)$ denotes the number of blocks of $G$ containing $\xi$) and since $|X_1\cap B|\geqq q(\omega-1)+1$ then $m^*>0$. Therefore

$$|X_1|\leqq\alpha q+\beta+\frac{1}{m^*}\Big(\sum_{i=1}^{f}(\alpha q+\beta-|S_i|)\leqq d(\alpha,\beta).$$

Now consider $\gamma,\delta\in X_1$, $F(\gamma,\delta)=\varrho$, and $\operatorname{cl}\gamma=\{B_1,\dots,B_q\}$. Clearly $|B_i\cap X_2|\geqq$ $\geqq k-\alpha q-\beta$ and then

$$|X_1|\leqq n-q(k-\alpha q-\beta)+j-\varrho+1. \ ▮$$

The theorem below shows how to use the above lemmas.

**Theorem 4.4.** *Let* $q>5$, $j^2\geqq q-1$, $k\geqq(j+1)(q^2-q+1)$. *Then*

$$|F|\geqq L(n,k,2)+q=q^2+q.$$

**Proof.** First suppose that $|X_1\cap B|>(j+1)q$ for some $B\in F$. Lemma 4.1 ($i=1$) gives $F(\xi)\geqq q+2$ for every $\xi\in X\setminus B$, and then

$$|F|\geqq(nq+2(n-k))/k\geqq q^2+q.$$

Thus, if $X_1\subseteqq B$ for some $B\in F$ then $|X_1|\leqq(j+1)q$ and then

$$|F|\geqq\lceil(n(q+1)-(j+1)q)/k\rceil\geqq q^2+q.$$

Now suppose that $|B\cap X_1|\leqq(q-2)j$ for every $B\in F$. This gives $|X_1|\leqq$ $\leqq q(q-2)j-q+1$ and consequently $|F|\geqq q^2+q$.

Finally suppose that $B$ is a maximal block of $F$, $|B\cap X_1|\geqq(q-2)j+1$ and there exists a centre $\xi\in X_1\setminus B$. Let $\operatorname{cl}\xi=\{B_1,\dots,B_q\}$. Since $k>(j+1)q$ we can suppose that $B_q\cap B\cap X_1=\emptyset$ (see the proof of Lemma 4.2). We distinguish two cases.

(a) $|B_h\cap B\cap X_1|=j+1$ for some $h\leqq q$. Here $|X_1|\leqq(j+1)(2q-3)$ due to Lemma 4.2 and $|F|\geqq\lceil(n(q+1)-(j+1)(2q-3))/k\rceil\geqq q^2+q$.

(b) $|B_i\cap B\cap X_1|\leqq j$, $i=1,\dots,q-1$. Therefore $|B\cap X_1|\leqq(q-1)j$. It is easy to see that we can make use of Lemma 4.3 (the case $|X_1|\leqq d(\alpha,\beta)$) substituting $m^*$ by $q-4$, $f$ by $q-1$, $\alpha q+\beta$ by $j(q-1)$, $\tau$ by $\omega$. Thus $|X_1|\leqq\{(2q-5)(q-1)j-(q-1)\omega\}/(q-4)$ and again $|F|\geqq q^2+q$. ▮

**Corollary 4.5.** *Let* $n=q^2r-h$, $q>5$, $0\leqq h\leqq i$, $i\geqq q-1+\sqrt{q}-1$,

$$k=qr\geqq(i-q+2)(q^2-q+1),$$

*and let q be a prime power. Then*

$$C(n,k,2)=q^2+q.$$

**Proof.** We have $C(q^2r-h, qr, 2) \geqq C(q^2r-i, qr, 2)$. Since $(i-q+1)^2 \geqq q-1$, $q^2r-i-1=q(k-1)-(i-q+1)$, and $k \geqq ((i-q+1)+1)(q^2-q+1)$ we obtain

$$C(q^2r-i, qr, 2) \geqq q^2+q$$

due to Theorem 4.4. On the other hand there exists an affine plane AG $(2, q)$, whence

$$C(q^2r-h, qr, 2) \leqq C(q^2r, qr, 2) \leqq C(q^2, q, 2) = q^2+q. \quad \blacksquare$$

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